The contribution of the remaining area of Fig. 2 is obtained by means of Eq. (8) and the radius vector:

$$\rho_3 = a[-v\cos\omega + (1-v^2\sin^2\omega)^{1/2}] \tag{13}$$

Therefore, after substitution of Eq. (13) into Eq. (8) and integration, we have

$$T_3 = (qa/2\pi\lambda)\,[E(v)-v] \tag{14}$$

where E is the complete elliptic integral of the second kind of modulus v.

Adding Eqs. (7, 12, and 14) and multiplying by 2 yields the centroidal temperature. Dividing this temperature by the total heat flow rate and then normalizing this value yields the dimensionless centroidal resistance:

$$\lambda\sqrt{A}\,R_0 = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\left[v\,\ell n\tan\left(\frac{\pi}{4}+\frac{1}{2}\tan^{-1}\frac{1}{v}\right)\right.$$

$$\left.+E(\sin^{-1}v,v)+E(v)-v(1+v^2)^{1/2}\right] \tag{15}$$

Since $v=4/3\pi$, Eq. (15) reduces to $\lambda\sqrt{A}\,R_0=0.5456$. The ratio of the area average resistance to the centroidal resistance for this contact area is 0.8449.

L-Shaped Contacts

The desired L-shaped contact is generated by the removal of a rectangular area ($\xi\eta$) from the lower left-hand corner of a unit square area. The origin of the Cartesian coordinate system is located at the lower left-hand corner of the unit square. The centroidal and area average temperatures and the corresponding dimensionless resistances were obtained by varying ξ and η systematically. In this work, ξ was allowed to range from 0 to 0.5 with increments of 0.10, while η ran from 0 to 1.0 for each value of ξ, in increments of 0.10. In this manner, 50 different L-shaped contacts were examined.[4]

When $\xi=\eta=0.1$, the largest values of the dimensionless resistances were observed, these being $\lambda\sqrt{A}\bar{R}=0.4733$ and $\lambda\sqrt{A}\,R_0=0.5614$. The smallest values, $\lambda\sqrt{A}\,\bar{R}=0.4424$ and $\lambda\sqrt{A}\,R_0=0.5197$ were observed at $\xi=0.5$, $\eta=0.7$. The maximum value of $\bar{R}/R_0=0.8540$ occurred at $\xi=0.5$, $\eta=0.6$, while the minimum value of $\bar{R}/R_0=0.8321$ was noted at $\xi=0.5$, $\eta=0.9$. They differ from the value of 0.8400 by +1.67% and −0.95%, respectively.

Summary and Conclusion

We see that, in the range $0.4\leq\alpha\leq 2$, both constriction resistances, $\bar{R}$ and R_0, are relatively insensitive to the aspect ratio and, further, that these triangular shapes have constriction resistances that do not differ substantially from the constriction resistances of a circular contact area ($\lambda\sqrt{A}\bar{R}=0.4787$, $\lambda\sqrt{A}R_0=0.5642$). For $\alpha<0.4$ and $\alpha>2$, both resistances decrease, and these values no longer can be considered comparable to those of the circular contact area. We note that the ratio $\bar{R}/R_0$ is much less sensitive to the aspect ratio α over its entire range, and that the values agree closely with the ratio corresponding to the circular contact. The results of the analysis for the semicircular contact area are remarkable for two reasons:

1) The normalized resistances are less than 4% different.

2) These resistances are less than the corresponding resistances for the circular contact; that is to say, the nonsymmetric contact offers less constriction resistance than the symmetric contact area.

The L-shaped contact area results are even more remarkable because, although this contact has no symmetry, its constriction resistance is very close to that of the triangular and semicircular contact areas and smaller than that of the circular contact. Its ratio $\bar{R}/R_0$ is in remarkably good agreement with the results of the triangular and semicircular contact areas.

The results of this study allow us to conclude that one can estimate the area average temperature (or constriction resistance) of nonsymmetric contact areas by taking 84% of the centroidal temperature (or constriction resistance) with an error less than ±1.7%, provided that the contact area is not too asymmetric. The constriction resistance of nonsymmetric contact areas, whether based upon the area average or centroidal temperatures, will be less than the corresponding resistances for the circular contact. The normalized constriction resistance $\lambda\sqrt{A}\,R_0$ is approximately 5/9 for symmetric and nonsymmetric contacts when subjected to a uniform heat flux.

Acknowledgment

The authors thank the National Research Council for its financial support.

References

[1] Yovanovich, M. M., Burde, S. S., and Thompson, J. C., "Thermal Constriction Resistance of Arbitrary Planar Contact with Constant Flux," AIAA *Progress in Astronautics and Aeronautics: Thermophysics of Spacecraft and Outer Planet Entry Probes*, Vol. 56, edited by A. M. Smith, New York, 1977, pp. 127-140.

[2] Yovanovich, M. M., "Thermal Constriction Resistance of Contacts on a Half-Space: Integral Formulation," AIAA *Progress in Astronautics and Aeronautics: Radiative Transfer and Thermal Control*, Vol. 49, edited by A. M. Smith, New York, 1976, pp. 397-418.

[3] Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series, and Products*, Academic Press, New York, 1965.

[4] Burde, S. S., "Thermal Contact Resistance between Smooth Spheres and Rough Flats," Ph.D. Thesis, Dept. of Mechanical Engineering, Univ. of Waterloo, Ontario, March 1977.

Determination of Pole Sensitivities by Danilevskii's Method

James B. Nail,* Jerrel R. Mitchell,† and Willie L. McDaniel Jr.‡
Mississippi State University, Mississippi State, Miss.

I. Introduction

In control theory, a synonymous term to pole sensitivity is eigenvalue sensitivity. Several methods for calculating eigenvalue sensitivities have been presented.[1-5] In general, these methods either require an application of Leverrier's method[2] or require the determination of eigenrows and eigencolumns.[1,3-5] Although Leverrier's method has a theoretically sound basis, it suffers from truncation errors when implemented on a digital computer. From experience these authors have found that numerical results from Leverrier's method cannot be trusted for systems roughly greater than tenth order.§

The techniques utilizing eigenrows and eigencolumns are suitable if the sensitivities of only a few eigenvalues are sought. However, if the sensitivities of several eigenvalues are required, then the calculation of the needed eigenrows and eigencolumns can be a formidable task.

In this paper, an alternate approach for calculating sensitivities of poles and eigenvalues is presented. Danilevskii's

Received April 18, 1977; revision received June 9, 1977.
Index category: Analytical and Numerical Methods.

*Research Associate, Department of Aerospace Engineering.
†Associate Professor, Department of Electrical Engineering.
‡Professor, Department of Electrical Engineering.

§This conclusion was reached while using a UNIVAC 1106 with double precision arithmetic.

method is shown to be suitable for performing the necessary evaluations. The result is a rational function that can be used to evaluate the sensitivities for all distinct poles.

A major goal in this work was to develop a technique that was applicable to high-order systems. The technique presented in this paper has been routinely used to generate eigenvalue sensitivities for systems up to 26th order while employing a UNIVAC 1106. It appears that the technique can be used for much higher order systems.

II. Preliminaries

The characteristic equation of an $n \times n$ matrix A is

$$\Delta(\lambda) = |\lambda I - A| = 0 \tag{1}$$

where I is the identity matrix, $|\,|$ denotes the determinant operation, and λ is an eigenvalue. Also, suppose that the elements of A are continuous, differentiable functions of a parameter x. Then, $|\lambda I - A|$ is also a continuous, differentiable function of x. Taking the partial derivative of Eq. (1) with respect to x gives

$$\frac{\partial|\lambda I - A|}{\partial \lambda}\frac{\partial \lambda}{\partial x} + \frac{\partial |\lambda I - A|}{\partial x} = 0 \tag{2}$$

Assuming λ as distinct, Eq. (2) can be solved for $\partial\lambda/\partial x$, i.e.,

$$\frac{\partial \lambda}{\partial x} = -\frac{1}{\Delta'(\lambda)}\frac{\partial|\lambda I - A|}{\partial x} \tag{3}$$

where $\Delta'(\lambda) = \partial\Delta(\lambda)/\partial\lambda$.¶ It is obvious that Eq. (3) is a rational function in λ and is applicable to all distinct eigenvalues. The problem is the efficient determination of the polynomials $\Delta'(\lambda)$ and $\partial|\lambda I - A|/\partial x$. This is considered in the next section.

III. Evaluation

First, attention is focused on $\Delta'(\lambda)$.** All that is needed is the characteristic polynomial of A. A technique for determining the characteristic polynomial of any square matrix is Danilevskii's method which is summarized in the Appendix. This technique can be used to determine $\Delta(\lambda)$, and then the determination of $\Delta'(\lambda)$ is trivial.

The second term of Eq. (3) is not dealt with in such an easy manner. Using the chain rule for partial differentiation, this is written as

$$\frac{\partial|\lambda I - A|}{\partial x} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial|\lambda I - A|}{\partial b_{ij}}\frac{\partial b_{ij}}{\partial x} \tag{4}$$

where b_{ij} is the ijth element of $|\lambda I - A|$. This can be rewritten as

$$\frac{\partial|\lambda I - A|}{\partial x} \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}\frac{\partial b_{ij}}{\partial x} \tag{5}$$

where B_{ij} is the ijth cofactor of $|\lambda I - A|$.†† Danilevskii's method is used to evaluate a determinant of the form $|\lambda I - A|$ by transforming the coefficient matrix A to the Frobenius form. Although all of the cofactors in Eq. (5) do not have the form $|\lambda I - A|$, they may be converted to this form as we shall show.

The cofactors of diagonal elements have the correct form. Cofactors of off-diagonal elements are represented as equivalent determinants. An equivalent representation of a cofactor of an element is achieved by setting the element to unity and zeroing the other elements of the corresponding row or column. For example, if A is 3×3; then

$$[\lambda I - A] = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{vmatrix} \tag{6}$$

The cofactor of the $-a_{32}$ element of Eq. (6) can be equivalently represented as

$$B_{32} = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ 0 & 1 & \lambda \end{vmatrix} - \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ 0 & 0 & \lambda \end{vmatrix} \tag{7}$$

Both determinants in Eq. (7) can be evaluated by Danilevskii's method.

IV. Example

In order to illustrate the techniques presented in this paper consideration is given to a control system with the coefficient matrix

$$A = \begin{bmatrix} -3.0 & 0.0 & 0.0 & K \\ 1.0 & -1.0 & 0.0 & 0.0 \\ 0.0 & 2.0 & -2.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & -10.0 \end{bmatrix} \tag{8}$$

where the nominal value of K is -10. It is desired to calculate the sensitivity coefficients of the roots of the characteristic equation [Eq. (8)] as a function of K. Of course this entails the evaluation of (3).

Using Danilevskii's method (8) is transformed to

$$P = \begin{bmatrix} 0.0 & 0.0 & 0.0 & -80.0 \\ 1.0 & 0.0 & 0.0 & -116.0 \\ 0.0 & 1.0 & 0.0 & -71.0 \\ 0.0 & 0.0 & 1.0 & -16.0 \end{bmatrix} \tag{9}$$

The characteristic polynomial of both Eqs. (8) and (9) is

$$|\lambda I - A| = \lambda^4 + 16.0\lambda^3 + 71.0\lambda^2 + 116.0\lambda + 80.0 \tag{10}$$

Then

$$\Delta'(\lambda) = 4\lambda^3 + 48.0\lambda^2 + 142.0\lambda + 116.0 \tag{11}$$

Using the technique presented in Sec. III, it is easily seen that

$$\frac{\partial|\lambda I - A|}{\partial K} = \begin{vmatrix} \lambda+3.0 & 0.0 & 0.0 & 1.0 \\ -1.0 & \lambda+1.0 & 0.0 & 0.0 \\ 0.0 & -2.0 & \lambda+2.0 & 0.0 \\ 0.0 & 0.0 & -1.0 & \lambda \end{vmatrix} - \begin{vmatrix} \lambda+3.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & \lambda+1.0 & 0.0 & 0.0 \\ 0.0 & -2.0 & \lambda+2.0 & 0.0 \\ 0.0 & 0.0 & -1.0 & \lambda \end{vmatrix} \tag{12}$$

¶It should be noted that Eq. (3) is equivalent to Eq. (2) in Ref. 3.

**Throughout this discussion it is assumed that the sensitivity is desired about some nominal value of x.

††Equation (5) is the expanded form of Adj$[\lambda I - A]^*$ $(\partial A/\partial x)$ where Adj means adjoint and * means inner product.

After recognizing the associative coefficient matrices, Danilevskii's method was used to evaluate each of these determinants. The results is

$$\frac{\partial|\lambda I-A|}{\lambda K}=(\lambda^4+6\lambda^3+11\lambda^2+6\lambda+2)-(\lambda^4+6\lambda^3+11\lambda^2+6\lambda) \tag{13}$$

or

$$\frac{\partial|\lambda I-A|}{\partial K}=2.0 \tag{14}$$

Thus,

$$\frac{\partial\lambda}{\partial K}=\frac{-2}{4\lambda^3+48\lambda^2+142\lambda+116} \tag{15}$$

Evaluating Eq. (15) at the roots of the characteristic equation, the sensitivity coefficients can be obtained.

Appendix

Danilevskii's method is a recursive technique for transforming any square matrix to the Forbenius form, i.e.,

$$P=\begin{bmatrix} 0 & 0 & \cdots & 0 & -d_0 \\ 1 & 0 & \cdots & 0 & -d_1 \\ 0 & 1 & \cdots & 0 & -d_2 \\ \vdots & \vdots & & & \\ 0 & 0 & & 1 & -d_{n-1} \end{bmatrix} \tag{A1}$$

Since the transformation is similar, the characteristic polynomials of Eq. (A1) and the original matrix are the same. It is easily verified that the characteristic polynomial of Eq. (A1) is

$$\Delta(\lambda)=|\lambda I-P|=\lambda^n+d_{n-1}\lambda^{n-1}+d_{n-2}\lambda^{n-2}+\ldots+d_0 \tag{A2}$$

The recurrence process is as follows:

$$A_{k+1}=S_k^{-1}A_kS_k \qquad k=1,2,\ldots,n-1 \tag{A3}$$

The matrix to be transformed to the Frobenius form is A; thus, $A_1=A$. The matrix S_k is formed from the identity matrix by replacing the $(k+1)$th column by the kth column of A_k. The matrix, S_k^{-1} is formed from the identity matrix by replacing the $(k+1)$th column by the negative of the kth column of A_k divided by the $(k+1)$th element except for the $(k+1)$th element, which is the positive reciprocal.

The recurrence process above suggests matrix operations. However, these operations are so simple that the kth step of the process can be described as follows:

$$b_{ij}=\begin{cases} a_{ij}/a_{ik} & \text{for } i=k+1 \\ a_{ij}-a_{ik}b_{k+1,j} & \text{for } i\neq k+1 \end{cases} \tag{A4}$$

and

$$a'_{ij}=\begin{cases} \sum_{\ell=1}^{n} b_{i\ell}a_{\ell k} & \text{for } j=k+1 \\ b_{ij} & \text{for } j\neq k+1 \end{cases} \tag{A5}$$

where $i=1,2,\ldots,n$, and $j=1,2,\ldots,n$; a_{ij} are the elements of the matrix A_k; a'_{ij} are the elements of the matrix A_{k+1}; b_{ij} are the elements of a "scratch-pad" matrix. The above are repeated $n-1$ times. For more details on implementing Danilevskii's method, see Refs. 6 and 7.

Acknowledgment

This work was partially supported by NASA, MSFC, Contract No. NAS8-31568.

References

[1] Van Ness, J. E., Boyle, J. M., and Imad, F. P., "Sensitivities of Large, Multi-Loop Control Systems," *IEEE Transactions on Automatic Control,* Vol. AC-10, July 1965, pp. 308-315.

[2] Morgan, B. S., Jr., "Sensitivity Analysis and Synthesis of Multivariable Systems," *IEEE Transactions on Automatic Control,* Vol. AC-11, July 1966, pp. 506-512.

[3] Reddy, D. C., "Evaluation of the Sensitivity Coefficient of an Eigenvalue," *IEEE Transactions on Automatic Control,* Vol. AC-12, Dec. 1967, p. 792.

[4] Rogers, L. C., "Derivatives of Eigenvalues and Eigenvectors," *AIAA Journal,* Vol. 8, May 1970, pp. 943-944.

[5] Rudisill, C. S. and Chu, Yee-Yeen, "Numerical Methods for Evaluating the Derivatives of Eigenvalues and Eigenvectors," *AIAA Journal,* Vol. 13, June 1975, pp. 834-837.

[6] Fadeev, D. K. and Fadeeva, V. N., *Computational Methods of Linear Algebra,* Freeman, San Francisco, Calif., 1963.

[7] Mitchell, Jerrel R., McDaniel, Willie L., and Nail, James B., "Control System Design by Pole Encouragement," Final Report, Contract No. NAS8-31568, NASA, MSFC, Sept. 3, 1976, pp. 68-76.

Contaminants in a Gasdynamic Mixing Laser

P. Hoffmann,* H. Hügel,† and W. Schall†
DFVLR, Institut für Technische Physik, Stuttgart, West Germany

Introduction

THE great potential of the gasdynamic mixing laser (GDML) as a high-energy system has been demonstrated by several groups, e.g., Refs. 1-4, to list but some of the more recent work. The basic idea that lies behind this conception is to produce separately thermally excited nitrogen and to inject cold carbon dioxide into a downstream region of lower temperature. By this means, advantage is taken of the facts that 1) the stagnation temperature can be raised to values far higher than the dissociation limit of CO_2, and 2) the freezing efficiency of a pure N_2-flow is extremely high. As a consequence, the available vibrational energy may be increased by approximately an order of magnitude as compared with typical values of premixed gasdynamic lasers (GDL).

In the work pertaining to the GDML, the nitrogen so far has been heated either in arc-heaters or in shock tubes. Both methods are useful and convenient in the course of basic investigations but may not be adequate in some applications. In those cases, a production of the hot N_2 by chemical means would be preferable. On the other hand, each combustion process yields products in addition to N_2, which might have adverse effects upon the molecular kinetics involved in the GDML.

The effect of some contaminating additives like O_2, NO, CO, SO_2, and H_2 on small signal gain and laser power of a small-scale GDML has been investigated experimentally and is reported in this Note.

Experiments and Discussions

The experimental apparatus used in this work is described in more detail in Ref. 4. Briefly, the N_2 is heated in a d.c. arc-

Received May 3, 1977; revision received June 16, 1977.
Index categories: Lasers; Thermochemistry and Chemical Kinetics.
*Research Scientist.
†Research Scientist, Member AIAA.